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Potentialities of Lunar Laser Ranging for Measuring Tectonic Motions

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Potentialities of lunar laser ranging for measuring tectonic motions

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The lunar laser-ranging system at McDonald Observatory, Texas, is currently attaining accuracies of ± 15 cm, and ± 3 cm appears feasible.

Numerical error analyses containing 97 parameters (35 of them for error sources) indicate that the ± 3 cm system would measure station motions to ± 1 cm/year accuracy within a year for east–west motions and within about 3 years for north–south. Furthermore, the correlations of station motions with other parameters are low, so that it is unlikely that the estimate is too optimistic because of modelling inadequacies of the error analysis.

1. Introduction

Laser ranging to the Moon has been carried out at the McDonald Observatory in west Texas since September 1969 under the direction of E. C. Silverberg. Since the installation of a new calibration system in December 1971, the accuracy has been close to the design goal of ± 0.15 m, as evidenced by r.m.s. fits of three parameters (essentially the mean location of the Moon for that night) to some series of observations lasting more than 7 h. Currently (May 1972) observations are made regularly of all three Apollo retroreflectors when the Moon is close to the meridian, and of the Apollo 15 retroreflector about 3 h before and after meridian passage. Measurements are successful on an average of 15 nights a lunation (see Silverberg & Currie 1971, for a description of the operation).

Analyses of the data (principally by P. L. Bender, J. D. Mulholland and J. G. Williams) do not yet obtain fits approaching the ± 0.15 m accuracy. For data spans of 4 months the r.m.s. residuals are about ± 2 m; of 2 years, about ± 10 m. It is almost certain these discrepancies are due to inadequacies or errors in the mathematical modelling and will be worked out in time (Mulholland *et al.* 1972).

Research in atmospheric refraction has reached the conclusion that with measurements of atmospheric pressure at the site, error from this source should be less than 2 cm (Bender & Kirkpatrick 1972). Hence it has been decided to instrument the second station to be installed at Mt Haleakela on Maui Island in the Hawaiians with a laser and detector of comparable accuracy. This station should be in operation late in 1973. It is also hoped that French and Soviet stations will be in regular operation by 1974.

It is fairly evident that it is most desirable to add stations in the southern hemisphere. Such stations would give better coverage of the lunar orbit throughout the month, as well as giving better geometry for determination of the variations in rate and direction ('wobble') of the Earth's rotation. The main purpose of this paper is to report on numerical error analyses made to test the improvement obtained by adding a southern hemisphere telescope, as well as to estimate the accuracy with which parameters of scientific interest can be determined. We shall examine especially the ability of the system to detect horizontal drifts of the ranging stations. A general discussion of the use of the laser ranging for geodesy is also given by Faller *et al.* (1972).

2. Numerical error analyses

These analyses were made at various times over the period 1966 to 1971. The general procedure was to construct a series of hypothetical range observations. For each observation, partial derivatives with respect to each of the parameters were formed. These partial derivatives were used to pre- and post-multiply an assumed weight and then increment a normal equation coefficient matrix. At the end of a specified duration the normal matrix was inverted to obtain a covariance matrix.

2.1 *Parameters*

Seventeen types of parameters were allowed for in the program. These seventeen types can be grouped into four classes, according to their scientific interest:

Geometrical and orbital: station coordinates, retro-reflector coordinates, orbital constants-of-integration, obliquity, and scale.

Geophysical: station drift, polar wobble, rotation variations, Earth tide, and orbital acceleration.

Selenophysical: forced physical librations, free librations, and Moon tide.

Systematic error sources: fixed bias, zenith-distance bias, sun-angle bias, and arbitrary periodic biases.

All but two types (orbital acceleration, free librations) were included in the calculations. For six types (orbital constants, scale, and the four biases) *a priori* standard deviations were prescribed. For two types (wobble, rotation) only a few frequencies of their large spectra were included. In the discussion that follows the mathematical derivation of partial derivatives will not be given where it can be readily obtained from Kaula (1966).

2.1.1. *Geometrical and orbital parameters*

The effect of the Earth's rotation is to impose on the range from a given station a 1 cycle/25 h modulation. The phase of this modulation depends on the station longitude, while the amplitude depends on its distance off the rotation axis. The third component of station position, parallel to the rotation axis, will affect the range only through a slowly varying parallactic offset. Hence we should expect the error ellipsoid for a station to have its shortest axis in the east–west direction and its longest parallel to the rotation axis.

Because of the Moon's synchronous rotation, there will not be any large modulation effect to help discriminate components of the retroreflector position. We should expect the component toward the Earth to be well-determined (although highly correlated with Earth–Moon distance), but the two components at right angles thereto to be more weakly determined, and the parallel-to-axis component to be correlated with the similar component in the terrestrial station positions.

The orbital constants-of-integration assumed were mean Kepler elements at the start of the simulation. The partial derivatives were based on a secularly precessing Kepler ellipse (Kaula 1966, pp. 67–73), *plus* the leading periodic solar perturbations. The latter addition was made in response to the belief of P. L. Bender (personal communication, 1969) that to discriminate fully between the different parameters the leading terms of the perturbation spectrum had to be used. Because of the possible weakness of the ranging in determining orbit orientation, *a priori* standard deviations based on the occultation and meridian observations (Brouwer & Watts 1948) were used.

The obliquity expresses the orientation of the Earth-fixed system to the eliptic, to which the orbit is referred. The two longitudinal orientation angles are not included here because one (the equinox) must be fixed to constitute the reference and the other (the Greenwich Sidereal Time) is indistinguishable from station longitudes. The scale relates the dynamical system of the orbit to the geometric system of the ranging stations; its determination is equivalent to determining GM, so an *a priori* standard deviation based on the determination thereof from space probe tracking (Anderson, Efron and Wong 1970) is used.

2.1.2. *Geophysical parameters*

The geophysical interest is not in the absolute positions of the ranging stations, but in their changes. Hence for station drift explicit parameters are included, two for each station: horizontal motion is assumed. The east–west component derives from the change in longitude, and therefore should be most strongly determined. The north–south component derives mainly from the change in distance off the rotation axis, and therefore should be more strongly determined in high latitudes than in low, but nowhere as well as the east–west.

Polar wobble has a broad spectrum, with peaks at the Chandler (1 cycle/437 days), annual, semi-annual, plus possibly monthly and bi-monthly frequencies. The principal features which a lunar ranging system might discriminate in a year or so would be the frequencies of 1 cycle/month or higher. Hence for the present analysis four frequencies were taken: 1 cycle/13.251, 13.665, 14.106, and 27.330 days. 13.665 days is the bimonthly period; 13.251 and 14.106 days are the adjacent bands which observation over 437 days should discriminate. The same periods were taken for rotation variation, so a total of 24 parameters are required (sine and cosine coefficient for four frequencies of three components).

Earth tides cause a few tens of centimetres oscillation in station height. These were included in the form of a separate Love number h for each station.

Tidal acceleration of the Moon's orbit is about 3 cm radial per year and -180 cm along-track per year squared. This parameter was not included in the current analyses.

2.1.3. *Selenophysical parameters*

For the forced physical librations of the Moon, partial derivatives based on the leading 12 terms (Eckhardt 1965) were used. Only two parameters related to the moment-of-inertia differences were included in the solution: the ratios $\beta = (C-A)/B$ and $\gamma = (B-A)/C$. The third harmonics of the lunar gravitational field, which have short-periodic effects on the order of 2 m were not included, nor was the elasticity of the moon, which has an effect of about 40 ± 10 cm.

Allowance for free libration was included in the program; however, it was not included in any solution.

Lunar tides, which have a few tens of centimetres amplitude, were parameterized as the Love number h.

2.1.4. *Systematic error parameters*

The following form was assumed for the bias B of the system

$$B = B_0 + B_z \sec z + B_s \cos^6 \tfrac{1}{2}\psi + \sum_i (C_i \cos k_i t + S_i \sin k_i t), \qquad (1)$$

where z is the zenith angle of the Moon and ψ is the Moon–Sun angle. The *a priori* standard deviations assumed were 8 cm for B_0, 5 cm for B_z, 6 cm for B_s, and 1 to 6 cm for C_i and S_i. The frequencies k_i ranged from 1 cycle/13.61 days to 1 cycle/2190 days. Sixteen of these arbitrary periodic biases were included, so the total number of bias parameters was 35.

2.2 *Observational constraints*

Quantities specified in the input include: ranging station locations (Texas: 31° N, 256° E; Hawaii: 20.4° N, 204° E; plus hypothetical stations elsewhere); lunar retroreflector locations (2° N, 24° E; 2° S, 18° W; 26° N, 3° E); the duration in days of the ranging (normally 437); the interval in days between ranging attempts (normally 1); the number of ranges per attempt (normally 3 per day); the maximum zenith angle (normally 60°); the minimum Sun–Moon angle (normally 60°, owing to lack of guiding features on the new Moon); and the percentage of clear weather days (normally 40 to 60 %).

The variance S^2 of an observation is assumed to have the form

$$S^2 = S_0^2 + [S_z \sec z]^2 + [S_s \cos^6 \tfrac{1}{2}\psi]^2. \tag{2}$$

Values assumed were 10 cm for S_b, 10 cm for S_z, and 12 cm for S_s.

2.3. *Computing procedure*

After read-in of input and computation of various conversions, the time is set at the initial date specified and a calculation started which is over three nested 'DO' loops: the outer for all dates over the specified duration, the second over all stations, and the innermost over all retroreflectors.

The only calculations peculiar to each date alone are the time (in days) t_x, the corresponding Greenwich sidereal time θ_x, and the lunar longitude L_x. Then for each station the time t_c the Moon crosses its meridian is calculated by

$$t_c = (\dot{\theta} - \dot{L})^{-1} \bmod[(L_x - \theta_x - \lambda_i),\ 2\pi] + t_x, \tag{3}$$

where λ_i is station longitude.

A random number between 0 and 1 is chosen; if it exceeds the specified portion of clear days, the station is skipped. If the day is 'clear', the Moon's location at t_c is calculated and a determination made whether the Moon is within the maximum zenith angle and beyond the minimum Sun angle. If the Moon is within the required limits the variance is calculated by equation (2), and the DO loop over the retroreflectors is entered. For each station-retroreflector range, the partial derivatives $\partial O/\partial P_i$ with respect to each of the above described parameters are calculated. Then each element N_{ij} of the normal matrix is incremented by $(\partial O/\partial P_i)(\partial O/P_j)/S^2$.

If, as normal, off-meridian observations are specified, then their times are calculated as

$$t = t_c \pm (\dot{\theta} - \dot{L})^{-1} \cos^{-1}[(\cos z_x - \sin\delta \sin\phi_i)/(\cos\delta\cos\phi_i)], \tag{4}$$

where ϕ_i is station latitude and z_x is the maximum allowable zenith distance, and the same processes carried out as for t_c.

At the end of the specified duration, the inverse squares of any *a priori* standard deviations are added to the appropriate main diagonal elements of the normal matrix and the resulting matrix inverted to obtain a final covariance matrix.

2.4. *Results*

The standard deviations resulting from the various station sets tried are summarized in table 1. In cases where the actual number of parameters is more than one, the figure given is the mean for the type.

TABLE 1. STANDARD DEVIATIONS RESULTING FROM NUMERICAL ERROR ANALYSES OF ±15 cm LASER RANGING TO THE MOON FOR 437 DAYS

run no.	station locations	portion of clear days	observations per station
1	Texas, Hawaii	0.60, 0.40	352, 218
2	Texas, Hawaii, Australia	0.60, 0.40, 0.60	365, 247, 375
3	Texas, Hawaii, the Crimea	0.60, 0.40, 0.50	348, 202, 249
4	Texas, Hawaii, the Azores	0.60, 0.40, 0.50	326, 240, 314

parameter	unit	*a priori* std. dev.	standard deviations for run no. 1	2	3	4
geometrical and orbital						
sta. E–W	m	—	0.04	0.03	0.04	0.04
off-axis	m	—	0.30	0.15	0.20	0.18
par.-axis	m	—	0.55	0.44	0.45	0.45
refl. earthward	m	—	0.35	0.26	0.29	0.28
along-track	m	—	0.85	0.54	0.63	0.64
par.-axis	m	—	1.3	1.0	1.1	1.1
orbit node long.	10^{-9}	0.5×10^4	122	18	82	86
inc.	10^{-9}	0.5×10^4	16	4	12	12
perigee arg.	10^{-9}	10^5	122	19	82	86
S.-m. axis	$10^{-9} R_\oplus$	10^2	1.2	0.7	0.9	1.0
eccen.	10^{-9}	0.5×10^4	0.2	0.08	0.10	0.13
mean anom.	10^{-9}	0.15×10^5	2.6	1.4	1.7	1.7
scale	$10^{-9} R_\oplus$	10^2	4.3	2.6	3.1	3.6
obliquity	10^{-9}	—	24	12	16	16
geophysical						
sta. drift E–W	cm/year	—	4.5	4.1	4.4	4.4
N–S	cm/year	—	25	18	19	20
wobble 13.25d	cm	—	2.6	1.5	1.5	1.6
13.66	cm	—	3.8	1.8	1.9	2.1
14.11	cm	—	3.0	1.7	1.6	1.9
27.33d	cm	—	5.6	2.8	3.7	3.1
rotation 13.25d	10^{-6} s	—	18	10	9	11
13.66	10^{-6} s	—	34	20	22	24
14.11	10^{-6} s	—	21	13	12	14
27.33d	10^{-6} s	—	211	80	128	159
tidal Love no. *h*	—	—	0.60	0.29	0.49	0.44
selenophysical						
$(C-A)/B$	10^{-9}	—	0.5	0.33	0.43	0.41
$(B-A)/C$	10^{-9}	—	3.4	2.6	3.1	2.9
tidal Love no. *h*	—	—	0.19	0.14	0.16	0.16

It is evident from the table that a southern hemisphere station in Australia would be of considerably more benefit than another northern hemisphere station. The improvement is greatest first in the orientation parameters node, inclination, perigee, and obliquity, and secondly in the monthly variations of wobble and rotation. Both of these improvements are probably consequences of much better coverage of the Moon when it is in southerly latitudes, as well as an increased parallactic effect.

As anticipated, the system is much better at determining the east–west component of station motion than the north–south; it is more sensitive to phase shifts than to amplitude drifts. Runs for longer durations showed that the uncertainty in determining the drift rates is inversely proportional to the duration. Hence the ±15 cm ranging system should attain ±1 cm resolution for east–west motion in 5 years, and for north–south in 25 years.

The computation also generated a correlation matrix, which, for run no. 2, showed among the 97 parameters 14 correlations higher than 0.95; 7 of 0.85 to 0.95; 42 of 0.65 to 0.85; and 136 of 0.35 to 0.65. A high correlation does not detract from a low standard deviation *provided that the model used in the error analysis is an accurate representation of reality*. Since no model is perfect, high correlations are indicators of where the accuracy of results in reality are most likely to be poorer than in the error analysis because of inadequately modelled error sources. It is therefore worth while to examine all high correlations; we do so here for run no. 2: stations in Texas, Hawaii and Australia.

There were four sets of correlation coefficients with absolute values of more than 0.95. Three sets pertaining to coordinates were positive, indicating that the differences are measured much more accurately than the absolute values. These sets were the off-axis and parallel-to-axis components of station positions and the earthward component of retroreflector locations. In addition, there was the anticipated very high correlation between nodal longitude and perigee argument (negative, because of the small inclination).

There were two sets of correlations of about +0.9: between the parallel-to-axis components of retroreflector locations and between station Earth tides (the lunar Love no. h was assumed the same at all retroreflectors). The north–south components of station drifts had correlations of about +0.8, indicating that the differences are better determined than the absolute values:

$$\begin{aligned}\sigma(R\Delta\dot{\phi}) &= [2-2r]^{\frac{1}{2}}\,\sigma(R\dot{\phi})\\ &\approx [2-2\times 0.8]^{\frac{1}{2}}\times 20\\ &\approx 13\text{ cm.}\end{aligned} \tag{5}$$

There were nine other correlations of 0.7 or 0.8 among parameters of interest: between along-track components of retroreflector positions; between parallel-axis components of station and retroreflector positions; between station longitudes; between earthward coordinates of retroreflectors and lunar tidal Love number; between off-axis coordinates of stations and Earth tide; between along-track coordinates of retroreflectors and lunar tide; between orbital inclination and scale; between orbital mean anomaly and semi-major axis; and between orbital mean anomaly and a monthly variation in Earth rotation.

The only high correlation of an arbitrary bias with a parameter was −0.7 between a bias of monthly period and the along-track component of retroreflector location.

The only correlations of east–west station drift higher than 0.1 were −0.5 with station longitude and +0.2 with one component of polar wobble of 27.33-day period. The former is rather obvious and unavoidable; the latter presumably arises from non-uniform distribution of observations in a lunation. North–south station drift has correlations of 0.5 to 0.6 with the along-track components of retroreflector locations and −0.3 with the parallel-to-axis components. Again, the situation for differences in north–south drift should be better. Let x be a retroreflector coordinate, and a_1 and a_2 be north–south drifts. Then for the correlation of x with $\Delta a = a_1 - a_2$:

$$\begin{aligned} r\{x, \Delta a\} &= \frac{\langle x\Delta a\rangle}{[\langle x^2\rangle \langle(\Delta a)^2\rangle]^{\frac{1}{2}}} \\ &= \frac{\langle xa_1\rangle - \langle xa_2\rangle}{[\langle x^2\rangle\{2\langle a_i^2\rangle - 2\langle a_1\, a_2\rangle\}]^{\frac{1}{2}}} \\ &= \frac{\sigma_x \sigma_a[r\{x, a_1\} - r\{x, a_2\}]}{\sigma_x \sigma_a[2 - 2r\{a_1, a_2\}]^{\frac{1}{2}}} \\ &\approx \frac{0.6 - 0.5}{[2 - 2 \times 0.8]^{\frac{1}{2}}} \lesssim 0.15. \end{aligned} \tag{6}$$

Other runs were made to test such factors as changing the maximum allowable zenith distance, using only one retroreflector on the Moon, extending the duration of the simulation to 18.6 years (one nodal revolution), and omitting the periodic solar perturbations in the orbital partial derivatives. None of these changes produced any unexpected results. Using only one retroreflector has slight effect on determining parameters other than those pertaining to the Moon itself. The extended duration results in a proportionate improvement in the determination of all rates, as well as a better determination of orientation parameters. The omission of solar perturbations made negligible difference, for reasons discussed below.

3. Discussion

The purposes of this section are first to compare this error analysis with others, and secondly to attempt the jump from these mathematical modellings to a judgement about reality.

3.1. *Comparison with other analyses*

The most similar analysis to that described herein is by Fajemirokun (1971). An error analysis has also been made by P. L. Bender; it is not yet published, but discussion based thereon appears in papers such as Bender *et al.* (1971) and Faller *et al.* (1972).

Although the analysis of Fajemirokun is similar in technique, in that a hypothetical series of observations is used to increment a normal matrix, and provision is made for *a priori* variances, the analysis is significantly different in the selection of parameters and the assumed observing régime. Fajemirokun does not include orbital constants-of-integration, station drift, wobble and rotation variations, or Earth and lunar tides. He does include for both the Earth and Moon all three Euler angles and their rates at epoch, and assumes both bodies to be perfectly rigid. Also, three parameters related to the moment-of-inertia differences of the Moon are included, instead of the usual two. Observations were assumed to be made only at integer multiples of 0.5 day, and to have a duration of one year. No attempt was made to include systematic biases of zenith or Sun-angle dependent observation variances or to omit observations for unclear weather.

As should be expected, the standard deviations obtained were higher – although by less than a factor of 2, in the case of tracking station coordinates. The greatest difference was in much larger standard deviations for the parallel-to-axis components of retroreflector positions; by about a factor of 6. Why is not clear. About the same pattern of correlation coefficients for station and reflector coordinates was obtained, although generally higher.

The modelling of the orientation angles and their rates seems disproportionately elaborate, in view of the other omissions. Surely at least in the case of the Moon it would be more reasonable to assume for the present that the free librations are completely damped, since meteorite flux has been inadequate for the last couple of billion years, to give a perceptible probability of excitation.

The analysis of Bender is not a conventional normal matrix accumulation, but rather an estimate of the effects of systematic errors, assuming that they are distributed in a spectrum including bands overlapping the frequencies upon which determination of the parameters depends. He also assumed a random spectrum of polar wobble and rotational variations rich enough that, in effect, all off-meridian observations were required to determine them, leaving only meridian passage observations to determine parameters of the Moon and its orbit. To discriminate orbital parameters from retroreflector coordinates, Bender contends that several leading periodic terms in the solar perturbation of the lunar orbit should be included in the error analysis. In particular, this need applies to the separation of scale from the earthward reflector coordinate, and of mean anomaly from the along-track reflector coordinate. However, viewing the variation of range with time as a Fourier series, several frequencies which are generated by the unperturbed motion of the Moon in its eccentric orbit ('optical libration') and by the rotation of the Earth have amplitudes larger than any solar perturbation. The rotation terms are excluded by the limitation to meridian observations, which seems unduly pessimistic. For durations of more than a month or so, the secular changes in the Moon's orbit must be included and are more powerful in determining the action elements (semi-major axis, eccentricity, and inclination) associated with energy and angular momentum conservation.

Because of the aforestated differences in approach as well as differences in assumed frequency and duration of observations, the results of Bender cannot be compared directly to those in this paper, even though similar numerical results were obtained for station positions, orbital constants-of-integration, and geophysical parameters. Larger uncertainties were obtained for retroreflector positions and selenophysical parameters.

3.2 *General conclusions*

Since all the final standard deviations in table 1 are significantly smaller than *a priori*, a change in the observational uncertainty from ± 15 cm to ± 3 cm would divide all the numbers by five. Thus the analysis indicates that it would take the ± 3 cm system about 1 year to get ± 1 cm/year for east–west motion and 5 years for north–south; or (using equation (5)) 3 years for differences in north–south motion.

The principal reasons for this analysis being over optimistic are: (1) the assumption of randomness of the weather from day to day, and (2) the incompleteness of the wobble and rotation spectrum. For a given number of observations and a white error spectrum, a random series is actually the best possible distribution of the observations (Shapiro & Silverman 1960). A more realistic modelling would be a combination of systematic seasonal variations with a stochastic ensemble in which there would be a spectrum of decay times for randomization. This observing schedule model should also include shutdowns for instrumental repairs, etc.

The omitted wobble and rotation spectra would have the effect of greatly increasing the noise level. How much is hard to estimate, since it requires extrapolation upward in frequency from competing geophysical hypotheses of the causes for presently observed variations. Of specific spectral lines, it is difficult to imagine a term more likely to distort the station drift determination than the monthly other than the diurnal, which seems too short a time within which to budge the Earth perceptibly.

An objective error analysis comparison of laser ranging with very long baseline interferometry (v.l.b.i.) is difficult. Given similar observation error models, the laser ranging to the Moon will show to advantage compared to v.l.b.i. because it measures range rather than range difference, and hence will have many more opportunities for observations high in the sky. This advantage depends on the lunar orbit calculation being of negligible error for durations up to 20 h; the closeness of the three-parameter fit to 7 h tracking already attained indicates the accuracy is feasible. The questions are then whether v.l.b.i. can attain a smaller observational error by radiometry of water vapour (Schaper, Staelin & Waters 1970) or can compensate through the availability of a sufficient number of point sources.

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